Point classification of the second order ODE's by Ruslan Sharipov and its application to Painleve equations

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Abstract. This is an review on the point classification of second order ODE's by Ruslan Sharipov. His works were published in 1997-1998 at the Electronic Archive at LANL and undeservedly forgotten. Last chapter is an application of this classification to the investigation of Painleve equations.

1 Introduction

Let us consider the following second order ODE:

$$y'' = P(x,y) + 3\,Q(x,y)y' + 3\,R(x,y)y'^2 + S(x,y)y'^3. \tag{1}$$

General point transformations

$$\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y) \tag{2}$$

preserve the form of equation (1):

$$\tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\,\tilde{Q}(\tilde{x},\tilde{y})\tilde{y}' + 3\,\tilde{R}(\tilde{x},\tilde{y})\tilde{y}'^2 + \tilde{S}(\tilde{x},\tilde{y})\tilde{y}'^3. \tag{3}$$

Let us consider two arbitrary equations (1) and (3). The problem of existence of the point transformation (2) that connects these equations is called *the Equivalence Problem.* For the arbitrary equations (1) the explicit solution of the equivalence problem is rather complicated, see [3], [4].

The main approach that allows to solve the equivalence problem is based on the Invariant Theory. *Invariant* is a certain function depending on $(x,\,y)$ that is unchanged under (2):

$$I(x,y) = I(\tilde{x}(x,y), \tilde{y}(x,y)).$$

Invariant Theory of equations (1) goes back to the classical works of R.Liouville [1], S.Lie [2], A.Tresse [3], [4], E.Cartan [5], [6] (Late 19th- and Early 20th-Century) and continues in the works of [7], [8], [9], [10], [11]. Background is described in papers [11], [12].

However, only the modern development of computer technology has allowed to make a real breakthrough. In the set of papers [13], [14], [15] Ruslan Sharipov managed to build the system of the (pseudo)invariants so that all their formulas are calculated explicitly via the coeffitions of the equation (1). On the base of this system he constructed the classification of the equations (1). It is more total than any previous classifications. Moreover in the each case the sequence of the invariants could be continued infinitely. This fact allows us to solve the equivalence problem for some equations. See works V.Kartak [16] and [17].

The present paper is a rewiew of important works [13], [14], [15]. Also added the additional subcases (subsection 5.8) that were not mentioned in these works. Last chapter is an application of this classification to the investigation of Painleve equations.

Pseudoinvariant of weight m is a certain function depending on $(x,\, y)$ that is transformed under (2) with factor $\det T$ (the Jacobi determinant) in the degree m:

$$J(x, y) = (\det T)^m \cdot J(\tilde{x}(x, y), \tilde{y}(x, y)), \quad T = \begin{pmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{pmatrix}.$$

Pseudotensorial field of weight m *and valence* (r, s) is an indexed set that transforms under change of variables (2) by the rule

$$F^{i_1\ldots i_r}_{j_1\ldots j_s} = (\det T)^m \sum_{p_1\ldots p_r} \sum_{q_1\ldots q_s} S^{i_1}_{p_1} \ldots S^{i_r}_{p_r} T^{q_1}_{j_1} \ldots T^{q_s}_{j_s} \tilde{F}^{p_1\ldots p_r}_{q_1\ldots q_s},$$

here $S = T^{-1}$. It is easy to check that only factor $(\det T)^m$ distinguishes the pseudotensorial field from the classical tensorial field.

The correlation between the (pseudo)invariants from works [14], [15] and the semiinvariants from works [5], [1] (as they were presented in [12]) shows in the section 6. Here and everywhere below notation $K_{i.j}$ denotes the partial differentiation: $K_{i.j} = \partial^{i+j} K / \partial x^i \partial y^j$.

2 Classification

From the functions P, Q, R and S – the coefficients of equation (1) – let us organize the 3-indexes massive by the following rule:

$$\Theta_{111} = P, \qquad \Theta_{121} = \Theta_{211} = \Theta_{112} = Q,$$
$$\Theta_{222} = S, \qquad \Theta_{122} = \Theta_{212} = \Theta_{221} = R.$$

As the 'Gramian matrixes' let us take the following couple:

$$d^{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \text{pseudotensorial field of the weight 1,}$$

$$d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \text{pseudotensorial field of the weight -1.}$$

Let us reise the first index

$$\Theta_{ij}^{k} = \sum_{r=1}^{2} d^{kr}\Theta_{rij}. \tag{4}$$

Under the change of variables (2) Θ_{ij}^{k} transforms "almost" as a affine connection. (The transformation rule is into the paper [13]).

Using Θ_{ij}^{k} as the affine connection let us construct the "curvature tensor":

$$\Omega_{rij}^{k} = \frac{\partial \Theta_{jr}^{k}}{\partial u^i} - \frac{\partial \Theta_{ir}^{k}}{\partial u^j} + \sum_{q=1}^{2} \Theta_{iq}^{k}\Theta_{jr}^{q} - \sum_{q=1}^{2} \Theta_{jq}^{k}\Theta_{ir}^{q}, \quad \text{here } u^1 = x,\ u^2 = y,$$

and the "Ricci tensor" $\Omega_{rj} = \sum_{k=1}^{2} \Omega_{rkj}^{k}$. The both objcts are not the tensors.

The following 3 indexes massive is the tensor:

$$W_{ijk} = \nabla_i \Omega_{jk} - \nabla_j \Omega_{ik}.$$

Here we use Θ_{ij}^{k} instead of the affine connection when made the covariant differentiation.

Using the tensor W_{ijk} let us construct the new pseudovectorial fields:

$$\alpha_k = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} W_{ijk} d^{ij} \qquad \text{pseudocovectorial field of weight 1,}$$

$$\beta_i = 3\nabla_i \alpha_k d^{kr}\alpha_r + \nabla_r \alpha_k d^{kr}\alpha_i \quad \text{pseudocovectorial field of weight 3.}$$

The coincident pseudovectorial fields are: $\alpha^j = d^{jk}\alpha_k$ of weight 2, $\beta^j = d^{ji}\beta_i$ of weight 4.

There are only 3 situations:

1. Pseudovectorial field $\boldsymbol{\alpha}$=0 *maximal degeneration case*,
2. Fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear: $3F^5 = \alpha^i\beta_i = 0$, *intermediate degeneration case*;
3. Fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-collinear: $3F^5 = \alpha^i\beta_i \neq 0$, *general case.*

3 Maximal degeneration case

The coordinates of the pseudovectorial field $\boldsymbol{\alpha}$ are $\alpha^1 = B$, $\alpha^2 = -A$, where

$$\begin{aligned} A &= P_{0.2} - 2Q_{1.1} + R_{2.0} + 2PS_{1.0} + SP_{1.0} - 3PR_{0.1} - 3RP_{0.1} - 3QR_{1.0} + 6QQ_{0.1}, \\ B &= S_{2.0} - 2R_{1.1} + Q_{0.2} - 2SP_{0.1} - PS_{0.1} + 3SQ_{1.0} + 3QS_{1.0} + 3RQ_{0.1} - 6RR_{1.0}. \end{aligned} \tag{5}$$

In this case the conditions $A = 0$ and $B = 0$ are hold. So the classical Lie Theorem is true: all equations are equivalent to

$$\tilde{y}'' = 0$$

by the point transformation (2). The dimension of the point symmetries group is equal to 8. See papers [1], [7] and many others.

4 General case

The pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-collinear, so their scalar product is not equal to 0.

The pseudoinvariant F of weight 5 is:

$$3F^5 = AG + BH, \qquad \text{where A and B from (5),} \tag{6}$$

$$G = -BB_{1.0} - 3AB_{0.1} + 4BA_{0.1} + 3SA^2 - 6RBA + 3QB^2,$$

$$H = -AA_{0.1} - 3BA_{1.0} + 4AB_{1.0} - 3PB^2 + 6QAB - 3RA^2.$$

As if $F \neq 0$, let's make two functions:

$$\varphi_1 = -\frac{\partial \ln F}{\partial x}, \quad \varphi_2 = -\frac{\partial \ln F}{\partial y}. \tag{7}$$

Then, using Θ^k_{ij} from (4), let's construct Γ^k_{ij} – an affine connection:

$$\Gamma^k_{ij} = \Theta^k_{ij} - \frac{\varphi_k\delta^k_j + \varphi_k\delta^k_i}{3}.$$

Let's construct two non-collinear vectorial fields:

$$\boldsymbol{X} = \frac{\boldsymbol{\alpha}}{F^2}, \qquad \boldsymbol{Y} = \frac{\boldsymbol{\beta}}{F^4}.$$

Connection components define the covariant differentiation of these fields:

$$\nabla_{\boldsymbol{X}}\boldsymbol{X} = \hat{\Gamma}^1_{11}\boldsymbol{X} + \hat{\Gamma}^2_{11}\boldsymbol{Y}, \qquad \nabla_{\boldsymbol{X}}\boldsymbol{Y} = \hat{\Gamma}^1_{12}\boldsymbol{X} + \hat{\Gamma}^2_{12}\boldsymbol{Y},$$

$$\nabla_{\boldsymbol{Y}}\boldsymbol{X} = \hat{\Gamma}^1_{21}\boldsymbol{X} + \hat{\Gamma}^2_{21}\boldsymbol{Y}, \qquad \nabla_{\boldsymbol{Y}}\boldsymbol{Y} = \hat{\Gamma}^1_{22}\boldsymbol{X} + \hat{\Gamma}^2_{22}\boldsymbol{Y}.$$

The quantities $\hat{\Gamma}^k_{ij}$ are the scalar invariants of the equation (1). In the paper [14] they were denoted:

$$I_3 = \hat{\Gamma}^1_{12}, \quad I_6 = \hat{\Gamma}^2_{21}, \quad I_7 = \hat{\Gamma}^1_{22}, \quad I_8 = \hat{\Gamma}^2_{22}.$$

By differentiation these invariants along vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ we get more invariants:

$$\boldsymbol{X}I_k = I_{k+8}, \qquad \boldsymbol{Y}I_k = I_{k+16}.$$

Repeating this procedure of differentiation along $\boldsymbol{X}$ and $\boldsymbol{Y}$, we can consruct the indefinite sequence of invariants. The explicit formulas for the basic four invariants:

$$I_3 = \frac{B(HG_{1.0} - GH_{1.0})}{3F^9} - \frac{A(HG_{0.1} - GH_{0.1})}{3F^9} + \frac{HF_{0.1} + GF_{1.0}}{3F^5} +$$
$$+ \frac{BG^2P}{3F^9} - \frac{(AG^2 - 2HBG)Q}{3F^9} + \frac{(BH^2 - 2HAG)R}{3F^9} - \frac{AH^2S}{3F^9},$$

$$I_6 = \frac{A_{0.1} - B_{1.0}}{3F^2} - \frac{AF_{0.1} - BF_{1.0}}{3F^3},$$

$$I_7 = \frac{GHG_{1.0} - G^2H_{1.0} + H^2G_{0.1} - HGH_{0.1} + G^3P + 3G^2HQ + 3GH^2R + H^3S}{3F^{11}},$$

$$I_8 = \frac{G(AG_{1.0} + BH_{1.0})}{3F^9} + \frac{H(AG_{0.1} + BH_{0.1})}{3F^9} - \frac{10(HF_{0.1} + GF_{1.0})}{3F^5} -$$
$$- \frac{BG^2P}{3F^9} + \frac{(AG^2 - 2HBG)Q}{3F^9} - \frac{(BH^2 - 2HAG)R}{3F^9} + \frac{AH^2S}{3F^9}.$$

The case of general position divides into three subcases:

1. into the infinite sequence of invariants I_k one can find two functionally independent ones;

2. invariants I_k are functionally dependent but not all of them are constants;

3. all invarians in the sequence I_k are constants.

Example. Equation 6.54 from the handbook E.Kamke [18]:

$$y'' = y^2 + 4yy' + y^2y'^2.$$

5 Intermediate degeneration case

In the case $F = 0$, but $A \neq 0$ or $B \neq 0$ the pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear.

Let us denote the new quantities φ_1 and φ_2. If $A \neq 0$ they equal to

$$\varphi_1 = -3\frac{BP + A_{1.0}}{5A} + \frac{3}{5}Q, \quad \varphi_2 = 3B\frac{BP + A_{1.0}}{5A^2} - 3\frac{B_{1.0} + A_{0.1} + 3BQ}{5A} + \frac{6}{5}R, \tag{8}$$

If $B \neq 0$:

$$\varphi_1 = -3A\frac{AS - B_{0.1}}{5B^2} - 3\frac{A_{0.1} + B_{1.0} - 3AR}{5B} - \frac{6}{5}Q, \quad \varphi_2 = 3\frac{AS - B_{0.1}}{5B} - \frac{3}{5}R. \tag{9}$$

They allow us to organize an affine connection Γ^k_{ij} using Θ^k_{ij} from (4) and a pseudoinvariant Ω of weight 1

$$\Gamma^k_{ij} = \Theta^k_{ij} - \frac{\varphi_k\delta^k_j + \varphi_k\delta^k_i}{3}, \qquad \Omega = \frac{5}{3}\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right). \tag{10}$$

The pseudoinvariant Ω in the case $A \neq 0$:

$$\begin{aligned}\Omega = {} & \frac{2BA_{1.0}(BP + A_{1.0})}{A^3} - \frac{(2B_{1.0} + 3BQ)A_{1.0}}{A^2} + \frac{(A_{0.1} - 2B_{1.0})BP}{A^2} - \\ & - \frac{BA_{2.0} + B^2P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \frac{3B_{1.0}Q + 3BQ_{1.0} - B_{0.1}P - BP_{0.1}}{A} + Q_{0.1} - 2R_{1.0}.\end{aligned} \tag{11}$$

The pseudoinvariant Ω in the case $B \neq 0$:

$$\begin{aligned}\Omega = {} & \frac{2AB_{0.1}(AS - B_{0.1})}{B^3} - \frac{(2A_{0.1} - 3AR)B_{0.1}}{B^2} + \frac{(B_{1.0} - 2A_{0.1})AS}{B^2} + \\ & + \frac{AB_{0.2} - A^2S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \frac{3A_{0.1}R + 3AR_{0.1} - A_{1.0}S - AS_{1.0}}{B} + R_{1.0} - 2Q_{0.1}.\end{aligned} \tag{12}$$

The rule of covariant differentiation of the pseudotensorial field was presented in [13]:

$$\nabla_k F^{i_1\ldots i_r}_{j_1\ldots j_s} = \frac{\partial F^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial u^k} + \sum_{n=1}^{r}\sum_{v_n=1}^{2}\Gamma^{i_n}_{kv_n}F^{i_1\ldots v_n\ldots i_r}_{j_1\ldots j_s} - \sum_{n=1}^{s}\sum_{w_n=1}^{2}\Gamma^{w_n}_{kj_n}F^{i_1\ldots i_r}_{j_1\ldots w_n\ldots j_s} + m\varphi_k F^{i_1\ldots i_r}_{j_1\ldots j_s}.$$

If the pseudotensorial field F has type (r, s) and weight m, then the pseudotensorial field ∇F has type $(r, s+1)$ and weight m.

Pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear, hence exists the coefficient N, it is the pseudoinvariant of weight 2, such that: $\boldsymbol{\beta} = 3N\boldsymbol{\alpha}$. Then

$$\xi^i = d^{ij}\nabla_j N, \qquad M = -\alpha_i\xi^i, \qquad \boldsymbol{\gamma} = -\boldsymbol{\xi} - 2\Omega\boldsymbol{\alpha}, \tag{13}$$

Here $\boldsymbol{\xi}$ – pseudovectorial field of weight 3; M – pseudoinvariant of weight 4; $\boldsymbol{\gamma}$ – pseudovectorial field of weight 3.

The pseudoinvariant N in the cases $A \neq 0$ and $B \neq 0$, respectively, is:

$$N = -\frac{H}{3A}, \qquad N = \frac{G}{3B}. \tag{14}$$

The pseudoinvariant M in the case $A \neq 0$:

$$M = -\frac{12BN(BP + A_{1.0})}{5A} + BN_{1.0} + \frac{24}{5}BNQ + \frac{6}{5}NB_{1.0} + \frac{6}{5}NA_{0.1} - AN_{0.1} - \frac{12}{5}ANR. \tag{15}$$

And in the case $B \neq 0$ is:

$$M = -\frac{12AN(AS - B_{0.1})}{5B} - AN_{0.1} + \frac{24}{5}ANR - \frac{6}{5}NA_{0.1} - \frac{6}{5}NB_{1.0} + BN_{1.0} - \frac{12}{5}BNQ. \tag{16}$$

In the case $A \neq 0$ the field $\boldsymbol{\gamma}$ is:

$$\begin{aligned} \gamma^1 &= -\frac{6BN(BP + A_{1.0})}{5A^2} + \frac{18NBQ}{5A} + \frac{6N(B_{1.0} + A_{0.1})}{5A} - N_{0.1} - \frac{12}{5}NR - 2\Omega B, \\ \gamma^2 &= -\frac{6N(BP + A_{1.0})}{5A} + N_{1.0} + \frac{6}{5}NQ + 2\Omega A. \end{aligned} \tag{17}$$

In the case $B \neq 0$ the field $\boldsymbol{\gamma}$ is:

$$\begin{aligned} \gamma^1 &= -\frac{6N(AN - B_{0.1})}{5B} - N_{0.1} + \frac{6}{5}NR - 2\Omega B, \\ \gamma^2 &= -\frac{6AN(AS - B_{0.1})}{5B^2} + \frac{18NAR}{5B} - \frac{6N(A_{0.1} + B_{1.0})}{5B} + N_{1.0} - \frac{12}{5}NQ + 2\Omega A. \end{aligned} \tag{18}$$

5.1 First case of intermediate degeneration: $M \neq 0$

If $M \neq 0$ (15), (16) then the pseudovectorial fields $\boldsymbol{\alpha}$ (5) and $\boldsymbol{\gamma}$ (17), (18) are non-collinear. Moreover, $N \neq 0$ (14). Let's concider the following expansion:

$$\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\gamma}$$

The basic invariants:

$$I_1 = \frac{M}{N^2}, \qquad I_2 = \frac{\Omega^2}{N}, \qquad I_3 = \frac{\hat{\Gamma}^1_{22}}{M}.$$

Here M, N and Ω are from (10)-(11), (12), (13)-(14). The explicit formula for $\hat{\Gamma}^1_{22}$:

$$\begin{aligned} \hat{\Gamma}^1_{22} =& \frac{\gamma^1\gamma^2(\gamma^1_{1.0} - \gamma^2_{0.1})}{M} + \frac{(\gamma^2)^2\gamma^1_{0.1} - (\gamma^1)^2\gamma^2_{1.0}}{M} + \\ &+ \frac{P(\gamma^1)^3 + 3Q(\gamma^1)^2\gamma^2 + 3R\gamma^1(\gamma^2)^2 + S(\gamma^2)^3}{M}. \end{aligned}$$

By differentiating invariants I_1, I_2 and I_3 along fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ we get new invariants:

$$I_{k+3} = \frac{\nabla_{\boldsymbol{\alpha}} I_k}{N}, \qquad I_{k+6} = \frac{(\nabla_{\boldsymbol{\gamma}} I_k)^2}{N^3}.$$

The first case of intermediate degeneration divides into three subcases:

1. into the infinite sequence of invariants I_k one can find two functionally independent ones, the algebra of point symmetries of the equation (1) is trivial;

2. invariants I_k are functionally dependent but not all of them are constants the algebra of point symmetries is 1-dimensional;

3. all invarians in the sequence I_k are constants, the algebra of point symmetries is 2-dimensional.

Example. Equation 6.45 from the handbook E.Kamke is from the first case of intermediate degeneration.

$$y'' = ay'^2 + by.$$

5.2 Second case of intermediate degeneration

If $M = 0$ (15), (16) then the pseudovectorial fields $\boldsymbol{\alpha}$ (5) and $\boldsymbol{\gamma}$ (17), (18) are collinear. Hence exists the coefficient Λ such that: $\boldsymbol{\gamma} = \Lambda\boldsymbol{\alpha}$. This pseudoinvariant of weight 1 in the cases $A \neq 0$ and $B \neq 0$, respectively, is:

$$\Lambda = -\frac{\gamma^2}{A}, \quad A \neq 0, \qquad \text{or} \qquad \Lambda = \frac{\gamma^1}{B}, \quad B \neq 0.$$

The explicit formulas:

$$\Lambda = -\frac{6N(AS - B_{0.1})}{5B^2} - \frac{N_{0.1}}{B} + \frac{6NR}{5B} - 2\Omega. \tag{19}$$

$$\Lambda = \frac{6N(BP + B_{1.0})}{5A^2} - \frac{N_{1.0}}{A} - \frac{6NQ}{5A} - 2\Omega. \tag{20}$$

Let's calculate the curvature tensor using the connections (10):

$$R^k_{qij} = \frac{\partial \Gamma^k_{jk}}{\partial u^i} - \frac{\partial \Gamma^k_{iq}}{\partial u^j} + \sum_{s=1}^{2} \Gamma^k_{is}\Gamma^s_{jq} - \sum_{s=1}^{2} \Gamma^k_{js}\Gamma^s_{iq}, \quad u^1 = x,\, u^2 = y.$$

And the pseudotensorial field of the weight 1:

$$R^k_q = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} R^i_{qij} d^{ij},$$

where λ_1 and λ_2 its eigenvalues.

Let's construct the pseudocovectorial field of the weight -1 in the case $A \neq 0$:

$$\omega_1 = -\frac{R^2_1}{A}, \qquad \omega_2 = \frac{\lambda_2 - R^2_2}{A}. \tag{21}$$

Their explicit formulas:

$$\omega_1 = \frac{12PR}{5A} - \frac{54}{25}\frac{Q^2}{A} - \frac{P_{0.1}}{A} + \frac{6Q_{1.0}}{5A} - \frac{PA_{0.1} + BP_{1.0} + A_{2.0}}{5A^2} - \\ - \frac{2B_{1.0}P}{5A^2} + \frac{3QA_{1.0} - 12PBQ}{25A^2} + \frac{6B^2P^2 + 12A_{1.0}BP + 6A^2_{1.0}}{25A^3}.$$

$$\omega_2 = \frac{6\Lambda + 3\Omega}{5A} + \frac{-5BP_{0.1} + 6BQ_{1.0} + 12RBP}{5A^2} - \frac{54}{25}\frac{BQ^2}{A^2} - \\ - \frac{2BB_{1.0}P + BA_{0.1}P + B^2P_{1.0} + BA_{2.0}}{5A^3} - \frac{12B^2PQ}{25A^3} + \\ + \frac{3BQA_{1.0}}{25A^3} + \frac{6BA^2_{1.0} + 6B^3P^2 + 12B^2A_{1.0}P}{25A^4}.$$

And in the case $B \neq 0$:

$$\omega_1 = \frac{R^1_1 - \lambda_2}{B}, \qquad \omega_2 = \frac{R^1_2}{B}. \tag{22}$$

Their explicit formulas:

$$\begin{aligned}\omega_1 = &-\frac{6\Lambda + 3\Omega}{5B} + \frac{5AS_{1.0} - 6AR_{0.1} + 12QAS}{5B^2} - \frac{54}{25}\frac{AR^2}{B^2} + \\ &+ \frac{2AA_{0.1}S + AB_{1.0}S + A^2S_{0.1} - AB_{0.2}}{5B^3} - \frac{12A^2SR}{25B^3} + \\ &+ \frac{3ARB_{0.1}}{25B^3} + \frac{6AB^2_{0.1} + 6A^3S^2 - 12A^2B_{0.1}S}{25B^4},\end{aligned}$$

$$\begin{aligned}\omega_2 = &\frac{12SQ}{5B} - \frac{54}{25}\frac{R^2}{B} + \frac{S_{1.0}}{B} - \frac{6R_{0.1}}{5B} + \frac{SB_{1.0} + AS_{0.1} - B_{0.2}}{5B^2} + \\ &+ \frac{2A_{0.1}S}{5B^2} - \frac{3RB_{0.1} + 12SAR}{25B^2} + \frac{6A^2S^2 - 12B_{0.1}AS + 6B^2_{0.1}}{25B^3}.\end{aligned}$$

Let's construct the field of the weight 1:

$$\boldsymbol{w} = N\boldsymbol{\omega} + \nabla\Lambda + \frac{1}{3}\nabla\Omega.$$

It is collinear to the pseudovectorial field $\boldsymbol{\alpha}$ (5), hence exists the proportionality factor $\boldsymbol{w} = K\boldsymbol{\alpha}$.

$$K = \frac{\Lambda_{1.0} + \Lambda\varphi_1}{A} + \frac{\Omega_{1.0} + \Omega\varphi_1}{3A} + \frac{N\omega_1}{A}, \qquad A \neq 0. \tag{23}$$

$$K = \frac{\Lambda_{0.1} + \Lambda\varphi_2}{B} + \frac{\Omega_{0.1} + \Omega\varphi_2}{3B} + \frac{N\omega_2}{B}, \qquad B \neq 0. \tag{24}$$

Let's make a pseudocovectorial field $\boldsymbol{\varepsilon}$ of the weight 1:

$$\boldsymbol{\varepsilon} = N\boldsymbol{\omega} + \nabla\Lambda.$$

After raising indices by means of skew-symmetric matrix d^{ij} we get the pseudovectorial field $\boldsymbol{\varepsilon}$ of the weight 2.

$$\varepsilon^1 = N\omega_2 + \Lambda_{0.1} + \varphi_2\Lambda, \qquad \varepsilon^2 = -N\omega_1 - \Lambda_{1.0} + \varphi_1\Lambda. \tag{25}$$

The fields $\boldsymbol{\varepsilon}$ (25) and $\boldsymbol{\alpha}$ (5) are non-collinear, so we are able to write

$$\nabla_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\varepsilon}.$$

$$\begin{aligned}\hat{\Gamma}^1_{22} = &\frac{5\varepsilon^1\varepsilon^2(\varepsilon^1_{1.0} - \varepsilon^2_{0.1})}{3N\Omega} + \frac{5(\varepsilon^2)^2\varepsilon^1_{0.1} - 5(\varepsilon^1)^2\varepsilon^2_{1.0}}{3N\Omega} + \\ &+ \frac{5P(\varepsilon^1)^3 + 15Q(\varepsilon^1)^2\varepsilon^2 + 15R\varepsilon^1(\varepsilon^2)^2 + 5S(\varepsilon^2)^3}{3N\Omega}.\end{aligned}$$

The pseudoscalar fields:

$$L = KN + \frac{5}{9}N + 3\Lambda\Omega + \frac{7}{9}\Omega^2 + 2\Lambda^2. \tag{26}$$

$$\begin{aligned}E =&\hat{\Gamma}^1_{22} - \frac{\nabla_{\boldsymbol{\varepsilon}}L}{N} + \frac{4\Lambda\nabla_{\boldsymbol{\varepsilon}}\Lambda}{N} + \frac{17\Omega\nabla_{\boldsymbol{\varepsilon}}\Lambda}{6N} + \frac{12L^2}{5N} - \frac{53L\Lambda\Omega}{5N} - \frac{48L\Lambda^2}{5N} - \frac{62L\Omega^2}{15N} - \frac{8L}{3} + \\ &+ \frac{48\Lambda^4}{5N} + \frac{106\Lambda^3\Omega}{5N} + \frac{16\Lambda^2}{3} + \frac{1163\Lambda^2\Omega^2}{60N} + \frac{137\Lambda\Omega^3}{18N} + \frac{50\Lambda\Omega}{9} + \frac{203\Omega^2}{108} + \frac{77\Omega^4}{135N} + \frac{20N}{27}.\end{aligned} \tag{27}$$

So we can make invariants, here Λ from (19), (20), Ω from (11), (12), N from (14), L from (26), E from (27):

$$I_1 = \frac{\Lambda^{12}}{\Omega^8 N^2}, \qquad I_2 = \frac{L^4}{N^2\Omega^4}, \qquad I_3 = \frac{E^6 N^4}{\Omega^{20}}.$$

In the second case of intermediate degeneration algebra of point symmetries of the equation (1) is 1-dimensional if and only if all invariants I_1, I_2, I_3 are identically constant. At the other cases the algebra is trivial.

5.3 Third case of intermediate degeneration

In this case $N \neq 0$ (14), $M = 0$ (15), (16), $\Omega = 0$ (11), (12), $\Lambda \neq 0$ (19), (20). Let's consider again the pseudocovectorial field $\boldsymbol{\omega}$ of the weight -1 from (21), (22). Upon raising indices by the matrix d^{ij} we get the vector field $\boldsymbol{\omega}$: $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$. As if $\Lambda \neq 0$, then $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are non-collinear and we can get the following relation:

$$\nabla_{\boldsymbol{\omega}}\boldsymbol{\omega} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\omega},$$

$$\begin{aligned}\hat{\Gamma}^1_{22} = {} & \frac{5\omega^1\omega^2(\omega^1_{1.0} - \omega^2_{0.1})}{\Lambda} + \frac{5(\omega^2)^2\omega^1_{0.1} - 5(\omega^1)^2\omega^2_{1.0}}{\Lambda} + \\ & + \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2\omega^2 + 15R\omega^1(\omega^2)^2 + 5S(\omega^2)^3}{6\Lambda}.\end{aligned}$$

In this case we define a new L and E, here K from (23), (24):

$$\begin{aligned} L &= K + \frac{5}{9} + \frac{2\Lambda^2}{N}, \\ E &= \hat{\Gamma}^1_{22} - \frac{\nabla_{\boldsymbol{\omega}} L}{N} + \frac{9L^2}{5N} - \frac{2L}{N} - \frac{12L\Lambda^2}{5N^2} + \frac{7\Lambda^2}{3N^2} + \frac{5}{9N} + \frac{63\Lambda^4}{20N^3}. \end{aligned} \tag{28}$$

And construct the invariants. Here L, E from (28), N from (14), Λ from (19), (20).

$$I_1 = \frac{L^8 N^6}{\Lambda^{12}}, \qquad I_2 = \frac{EN^3}{\Lambda^4}.$$

In the third case of intermediate degeneration algebra of point symmetries of the equation (1) is 1-dimensional if and only if both invariants I_1, I_2 are identically constant. At the other cases the algebra is trivial.

Example. Emden-Fowler equation 6.11 with $n = -3$ from the handbook E.Kamke [18]:

$$y'' = -\frac{ax^m}{y^3}.$$

5.4 Fouth case of intermediate degeneration

Here $N \neq 0$ (14), $M = 0$ (15), (16), $\Omega = 0$ (11), (12), $\Lambda = 0$ (19), (20), $K \neq -5/9$ (23), (24).

Let's consider again the vectorial field $\boldsymbol{\omega}$: $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$ from (21), (22). As if $\Lambda = 0$, then $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are collinear and we can define new scalar field Θ by the relationship $\boldsymbol{\omega} = \Theta\boldsymbol{\alpha}$.

$$\Theta = \frac{\omega_1}{A}, \quad A \neq 0, \qquad \Theta = \frac{\omega_2}{B}, \quad B \neq 0. \tag{29}$$

Let's consider the covariant differential $\boldsymbol{\theta} = \nabla\Theta$. This is pseudocovectorial field of the weight -2:

$$\theta_1 = \Theta_{1.0} - 2\varphi_1\Theta, \qquad \theta_2 = \Theta_{0.1} - 2\varphi_2\Theta. \tag{30}$$

The corresponding pseudovectorial field of the weight -1: $\theta^1 = \theta_2$, $\theta^2 = -\theta_1$. Let's calculate its contraction with $\boldsymbol{\alpha}$ (5):

$$L = -\frac{5}{9}\sum_{i=1}^{2}\alpha_i\theta^i. \tag{31}$$

And the following relation is true, here K from (23), (24)

$$L = K + \frac{5}{9}.$$

As if $L \neq 0$, the fields $\boldsymbol{\theta}$ (30) and $\boldsymbol{\alpha}$ (5) are non-collinear:

$$\nabla_{\boldsymbol{\theta}}\boldsymbol{\theta} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\theta},$$

$$\begin{aligned}\hat{\Gamma}^1_{22} = &-\frac{5\theta^1\theta^2(\theta^1_{1.0} - \theta^2_{0.1})}{9L} - \frac{5(\theta^2)^2\theta^1_{0.1} - 5(\theta^1)^2\theta^2_{1.0}}{9L} - \\ &- \frac{5P(\theta^1)^3 + 15Q(\theta^1)^2\theta^2 + 15R\theta^1(\theta^2)^2 + 5S(\theta^2)^3}{9L}.\end{aligned}$$

One more pseudoscalar field:

$$E = \hat{\Gamma}^1_{22} + \frac{27N}{5}\left(\Theta + \frac{5}{9N}\right)^3 - \frac{3}{4}\left(\Theta + \frac{5}{9N}\right)^2. \tag{32}$$

Invariant, here E from (32), N from (14), L from (31):

$$I_1 = \frac{E^6N^{12}}{L^{20}}.$$

In the fouth case of intermediate degeneration algebra of point symmetries of the equation (1) is 1-dimensional if and only if the invariant I_1 is identically constant. Otherwise the algebra is trivial.

5.5 Fifth case of intermediate degeneration

In this case $N \neq 0$ (14), $M = 0$ (15), (16), $\Omega = 0$ (11), (12), $\Lambda = 0$ (19), (20), $K = -5/9$ (23), (24). All equations (1) are equivalent to

$$y'' = \frac{1}{y^3}, \quad \text{another form} \quad y'' = -\frac{5}{12x}y' + \frac{4}{3}x^2y'^3.$$

The algebra of point symmetries is 3-dimensional.

5.6 Sixth case of intermediate degeneration

In this case $N = 0$ (14), $\Omega \neq 0$ (11), (12). The pseudovectorial fields $\boldsymbol{\omega}$, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$(21), (22) and $\boldsymbol{\alpha}$ (5) are non-collinear, so:

$$\nabla_{\boldsymbol{\omega}}\boldsymbol{\omega} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\omega},$$

$$\hat{\Gamma}^1_{22} = -\frac{5\omega^1\omega^2(\omega^1_{1.0} - \omega^2_{0.1})}{9\Omega} - \frac{5(\omega^2)^2\omega^1_{0.1} - 5(\omega^1)^2\omega^2_{1.0}}{9\Omega} - \\ - \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2\omega^2 + 15R\omega^1(\omega^2)^2 + 5S(\omega)^3}{9\Omega}.$$

Invariants, here K from (23), (24), Ω from (11), (12):

$$I_1 = L = \nabla_{\boldsymbol{\omega}} K - \frac{21}{25}K^2 - K,$$
$$I_2 = \Omega^2\hat{\Gamma}^1_{22} - \nabla_{\boldsymbol{\omega}} L - \frac{72}{625}K^3 + \frac{63}{50}K^2 + \frac{12}{25}KL - K - L.$$

In the sixth case of intermediate degeneration algebra of point symmetries of the equation (1) is 1-dimensional if and only if both invariants I_1, I_2 are identically constant. At the other cases the algebra is trivial.

Example. Equation 6.41 from the handbook E.Kamke [18]:

$$y'' = -y^3 - \frac{1}{2}y^2 + 3yy'.$$

5.7 Seventh case of intermediate degeneration

In this case $N = 0$ (14), $\Omega = 0$ (11), (12). The pseudovectorial fields $\boldsymbol{\theta}$ (30) and $\boldsymbol{\alpha}$ (5) are non-collinear, so:

$$\nabla_{\boldsymbol{\theta}}\boldsymbol{\theta} = \hat{\Gamma}^1_{22}\boldsymbol{\alpha} + \hat{\Gamma}^2_{22}\boldsymbol{\theta},$$
$$\hat{\Gamma}^1_{22} = \theta^1\theta^2(\theta^1_{1.0} - \theta^2_{0.1}) - (\theta^2)^2\theta^1_{0.1} + (\theta^1)^2\theta^2_{1.0} - \\ - P(\theta^1)^3 - 3Q(\theta^1)^2\theta^2 - 3R\theta^1(\theta^2)^2 - S(\theta)^3.$$

Let's denote the new pseudoscalar field and the invariant, here Θ from (29):

$$L = \hat{\Gamma}^1_{22} - \frac{1}{2}\Theta^2, \qquad I_1 = \frac{(\nabla_{\boldsymbol{\theta}} L)^4}{L^5}.$$

In the seventh case of intermediate degeneration algebra of point symmetries of the equation (1) is 2-dimensional if and only if the field $L = 0$; is 1-dimensional if and only if the field $L \neq 0$ and if the invariant I_1 is identically constant. At the other cases the algebra is trivial.

Example. Equation 6.5 from the handbook E.Kamke [18]:

$$y'' = ay^2 + bx + c.$$

5.8 Additional subcases of the intermediate degeneration

Let's organaize a new pseudovectorial field $\boldsymbol{\eta}$ and its scalar product with the field $\boldsymbol{\xi}$ (13):

$$\eta^i = d^{ij}\nabla_j M, \qquad Z = d_{ij}\eta^i\xi^j.$$

Z is a pseudoinvariant of the weight 7. Then the first case of the intermediate degeneration divides into the four subcases.

Subcase 1.1. $M \neq 0$, $\Omega \neq 0$, $Z \neq 0$.

Subcase 1.2. $M \neq 0$, $\Omega \neq 0$, $Z = 0$.

Subcase 1.3. $M \neq 0$, $\Omega = 0$, $Z \neq 0$.

Subcase 1.4. $M \neq 0$, $\Omega = 0$, $Z = 0$.

The seventh case of the intermediate degeneration also divides into the two subcases. As if Θ is pseudoinvariant of the weight -2.

Subcase 7.1. $N = 0$, $\Omega = 0$, $\Theta \neq 0$.

Subcase 7.2. $N = 0$, $\Omega = 0$, $\Theta = 0$.

5.9 Tree of the intermediate degeneration cases

The following diagramme illustrates the cases of the intermediate degeneration.

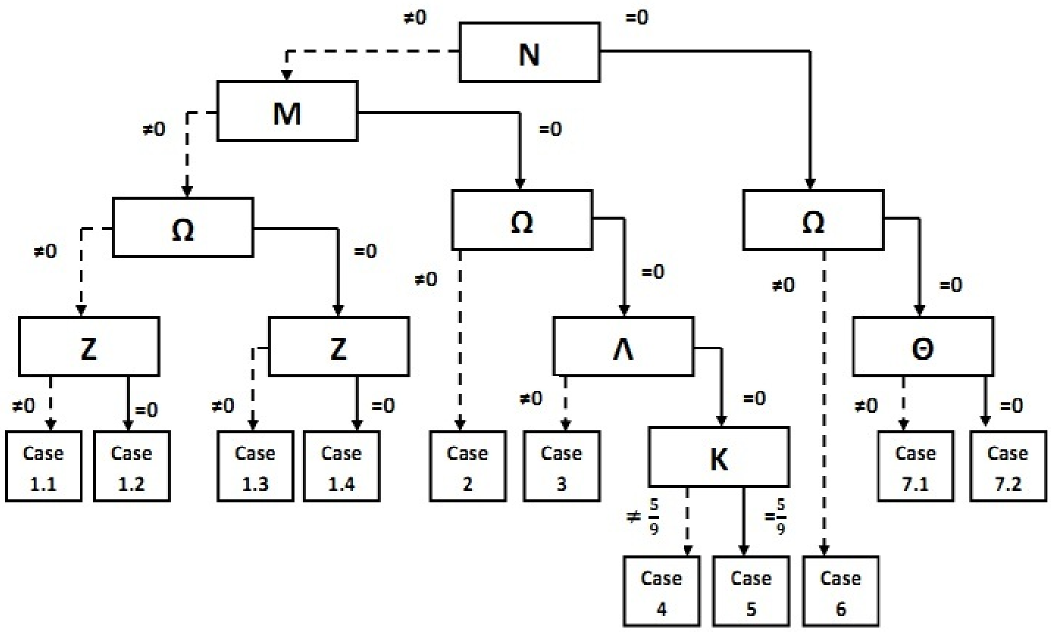

Рис. 1: Tree of the intermediate degeneration cases

6 Correlation between the semiinvariants

No doubt the main part of the pseudoinvariants have been known previously.

At the work E.Cartan [5] have adopted the following notations:

$$P = -a_4, \quad Q = -a_3, \quad R = -a_2, \quad S = -a_1, \quad A = -L_1, \quad B = -L_2.$$

At the work R.Liouville [1] were presented the semiinvariants ν_5, w_1, i_2 and the parameter R_1 (see rewiew in [12]). Here is a link between these quantities and the pseudoinvariants F, Ω, N and

the parameter H:

$$F^5 = \nu_5, \quad H = L_1(L_2)_x - L_2(L_1)_x + 3R_1, \quad \Omega = -w_1 - \frac{\nu_5 a_4}{L_1^3} - 4\frac{(L_1)_x R_1}{L_1^3}, \quad N = \frac{i_2}{3}.$$

Another pseudovectorial fields and pseudoinvariants for the first time were appeared in the papers [13], [14], [15].

7 Painleve equations

Let's determine the positions of the Painleve equations in the proposed classification scheme.

1. Equation Painleve I is in the case 7.1 of intermediate degeneration. The equivalence problem for this equation is effectively solved in paper [16].
2. Equations Painleve III-VI (with the exeption of the special cases!) are in the case 1.3 of intermediate degeneration.
3. Special cases.
   (a) Equation Painleve II is in the case 1.4 of intermediate degeneration. The equivalence problem for equation Painleve II is solved in papers [16], [17].
   (b) Equation Painleve III with 3 zero parameters is in the case 1.4 of intermediate degeneration. The equivalence problem for this equation is solved in paper [17].
   (c) Equation Painleve III with parameters (0,b,0,d) or (a,0,c,0) (they are equivalent) is in the case 1.4 of intermediate degeneration.
   (d) Equation Painleve V with parameters (a,b,0,0) is in the case 1.4 of intermediate degeneration.
   (e) Equation Painleve III with parameters (0,0,0,0) is in the case of maximal degeneration.
   (f) Equation Painleve V with parameters (0,0,0,0) is in the case of maximal degeneration.
   (g) Equation Painleve VI with parameters (0,0,0,1/2) is in the case of maximal degeneration.

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